

# Scattering along a complex loop in a solvable $\mathcal{PT}$ -symmetric model

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## Abstract

A non-unitary version of quantum scattering is studied via an exactly solvable toy model. The model is merely asymptotically local since the smooth path of the coordinate  $x$  is admitted complex in the non-asymptotic domain. At any real angular-momentum-like parameter  $\ell = \nu - 1/2$  the reflection  $R(\nu)$  and transmission  $T(\nu)$  are shown to change with the winding number (i.e., topology) of the path. The points of unitarity appear related to the points of existence of quantum-knot bound states.

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# 1 The concept of $\mathcal{PT}$ –symmetric toboggans

In the older physics literature the use of  $\mathcal{PT}$ –symmetric quantum Hamiltonians (i.e., of the operators  $H$  with the nonlinear symmetry property  $\mathcal{P}TH = H\mathcal{PT}$  where  $\mathcal{P}$  denotes parity while  $\mathcal{T}$  is complex conjugation which mimics time reversal) has been purely *formal* (cf., e.g., Remark 4 in Ref. [1] dated 1993). This notion just referred, implicitly or explicitly, to the use of the mathematical concept of Krein spaces (tractable, for our present purposes, just as the Hilbert spaces endowed with an auxiliary pseudometric  $\mathcal{P}$ , cf. [2]).

A thorough change of the *physical* paradigm has been inspired by Bender et al (cf., e.g., his own extensive account [3] of the history) who conjectured that the acceptance of the non-Hermitian  $\mathcal{PT}$ –symmetric quantum Hamiltonians may be perceived as “legal” and that it might, and does, thoroughly enrich our understanding of Quantum Theory as well as of the range of its applicability. Via a thorough analysis of the quasi-one-dimensional benchmark-model Hamiltonians

$$H^{(\delta)} = -\frac{d^2}{dx^2} - (ix)^{2+\delta}, \quad \delta \geq 0 \quad (1)$$

these authors clarified, in particular, that in spite of the manifest non-Hermiticity (in the “friendly but false” representation space  $L^2(\mathbb{R})$ ), these operators *may* generate the real, discrete and below bounded spectrum of (in principle, observable) energies. They have shown that one must only define these operators in an appropriate, “standard”, *ad hoc* Hilbert space  $\mathcal{H}^{(S)}$  (we use the terminology of papers [4, 5] where further details have been also discussed).

Naturally, the resulting  $\mathcal{PT}$ –symmetric version of quantum theory exhibits a number of immanent limitations and counterintuitive features. The most visible one may be illustrated via the benchmark-model Hamiltonian  $H^{(\delta)}$ . Its very definition requires a strongly counterintuitive,  $\delta$ –dependent construction of the physical Hilbert space  $\mathcal{H}^{(S)}$ . This space is being chosen as a rather exotic linear space of the square integrable functions  $f(x)$  which are defined along certain very specific complex curves  $x = x^{(\delta)}(s)$ ,  $s \in (-\infty, \infty)$ . In order to keep the spectrum real, these “unobservable coordinate” curves must necessarily be shaped as certain *complex*, left-right symmet-

ric, downwards-oriented hyperbolas in general. In addition, the  $|s| \rightarrow \infty$  asymptotes of these Hamiltonian-dependent curves must tend to parallel the negative imaginary axis at the sufficiently large exponents  $\delta$  [3].

Many traditional model-building concepts are put under serious questionmark in this setting. First of all, the “traditional” probability-density interpretation of the wave functions  $\psi(x)$  is lost. Our wave functions become defined along the above-mentioned complex curve  $x = x(s)$  of the “would-be coordinate”. Next, one must also speak about a coordinate-dependent kinetic energy and/or mass  $m = m(s)$  which may be complex. *In extremis*, this mass may even happen to acquire a *purely negative* real value again (cf., e.g., [6]).

One is forced to change (or at least to modify) also the traditional thinking about the possible link of the mathematical model to any experimental setup. An unexpectedly successful fulfillment of such a requirement has been achieved, fortunately, in many models where just a bound-state spectrum is to be studied [3]. In particular, an amazingly successful illustration of the underlying non-Hermitian-representation approach to bound state spectra may be found, under the nickname of “interacting boson models”, in nuclear physics [7].

The similar persuasive success is still lacking in the applications of the same philosophy to the scattering. In this alternative dynamical regime, the persistence of a number of very serious difficulties has been revealed and reported by Jones [8]. He noticed that as a consequence of the change of the traditional paradigm there emerge serious conceptual open questions in the very formulation of the setup of the scattering experiment. One of the most unpleasant obstacles emerged, for example, from an unexpected formal conflict between the “natural” requirements of the  $\mathcal{PT}$ -symmetry of the Hamiltonian  $H = p^2 + V$ , of the *local* nature of the force “prepared” at the short distances ( $V = V(x)$ ) and of the asymptotically free and causality-preserving nature of the incoming and outgoing asymptotic waves which are assumed measured at the large distances (i.e., mathematically speaking, of the scattering asymptotic boundary conditions).

The net conclusion of the latter study (cf. also the additional thorough analysis

[9] of the survival of the long-range-correlation puzzles in the “correct” Hilbert space  $\mathcal{H}^{(S)}$  was that the  $\mathcal{PT}$ –symmetric quantum scattering should only be interpreted as an effective theory where one assumes the explicit presence of certain “sinks” and “sources” in the space.

In the latter scenario, the unitarity of the scattering ceases to be guaranteed of course. In our subsequent study [10] we proposed to weaken or circumvent such a scepticism and to reinstall the unitarity of the quantum scattering by  $\mathcal{PT}$ –symmetric obstacles via the use of certain short-ranged nonlocalities in the potentials.

In another approach reported in Ref. [11] we proposed to try to move one step further. In place of using the rather complicated non-local integral-operator kernels  $V = V(x, x')$  we decided to keep the potential local,  $V = V(x)$ , and to introduce a new degree of freedom via a short-ranged “space-smearing” term attached to the mass ( $m \rightarrow m(x)$ ) or, better, to the energy term in the Schrödinger equation ( $E \rightarrow E \times W(x)$ ). In spite of the presence of the new term  $W \neq I$ , the resulting generalized Schrödinger equations (or, in our terminology, “Sturm-Schrödinger equations” [12]) still keep the trace of the phenomenological ambitions of  $\mathcal{PT}$ –symmetric models. On mathematical side, they proved also friendly and tractable by the standard “Hermitization” trick mediated, constructively, by the transition to the suitable *ad hoc* Hilbert space  $\mathcal{H}^{(S)}$ . This space has been shown to exist and to remain amenable to constructive considerations [13].

The climax of the story comes with the idea that the  $\mathcal{PT}$ –symmetric “Sturm-Schrödinger” differential equations (defined, say, along the real line of coordinates  $x(s) \equiv \mathbb{R}$ ) may be mathematically simplified via a suitable change of variables. This makes them equivalent to the “usual-Schrödinger” differential equations defined, anomalously, along the so called “tobogganic” curve of “would-be” complex coordinates  $x(s) = x^{(tobog.)}(s) \neq \mathbb{R}$ . Typically, the latter “tobogganic” contour connects several Riemann sheets of the wave function [14] so that the quantitative analysis of its spectrum (or scattering) becomes complicated.

Our previous letter [15] described an extremely elementary analytic (and, moreover, non-numerically solvable) tobogganic model of bound states. In our present

paper we just intend to complement this study by a parallel description of the same model in the scattering dynamical regime. Firstly, we shall demand that our tobogganic “would-be coordinate” curves  $x = x(s)$  remain *asymptotically real* (i.e., asymptotically observable, with  $x(s) \sim s \in \mathbb{R}$  at  $|s| \gg 1$ ). Secondly, our “over-schematic” toy-model potential will be taken over from Ref. [15]. In this manner, the exact solvability of the related scattering “Gedanken-experiment” will be retained.

## 2 Elementary model

The ordinary linear differential equation

$$-\frac{d^2}{dx^2} \psi(x) + \frac{\ell(\ell+1)}{x^2} \psi(x) + V(x) \psi(x) = E \psi(x) \quad (2)$$

is often encountered in the textbooks on quantum mechanics where it emerges, with  $x \in \mathbb{R}^+$ , as the so called radial part of the Schrödinger equation in  $D$  dimensions. Three years ago we proposed [15] an alternative quantum interpretation of Eq. (2) in which the path  $\mathcal{C} = \mathcal{C}^{(N)}$  of the “coordinate”  $x$  has been allowed complex, forming a loop-shaped curve,  $N$ –times encircling the branch point of  $\psi(x)$  at  $x = 0$ .

For the sake of simplicity the external potential itself has been assumed absent,  $V(x) = 0$ . The resulting bound-state solutions  $\psi(x)$  of Eq. (2) were then given in closed form called, due to its topological origin, “quantum knot”. It has been emphasized that the “quantum knot” solutions could find their natural and consistent physical interpretation, e.g., within the framework of the so called  $\mathcal{PT}$ –symmetric Quantum Mechanics (cf., e.g., the recent reviews [3, 16] for a more detailed exposition of this formalism).

In our present letter, as we already mentioned, we intend to complement the results of paper [15] by a parallel study of the problem of scattering. In a way paralleling and completing our previous study [15] we shall outline a few most interesting consequences of the acceptance of the  $\mathcal{PT}$ –symmetrization strategy in the case of the vanishing potential,  $V(x) = 0$ . The presence of just a “minimal” dynamical input will be compensated by the topologically nontrivial choice of the loop-shaped

paths  $x^{(tobog.)}(s)$  or rather  $\mathcal{C} = \mathcal{C}^{(N)}$  marked by a winding number  $N = 1, 2, \dots$ . A characteristic one-loop example of such a path with  $N = 1$  is displayed in Fig. 1.

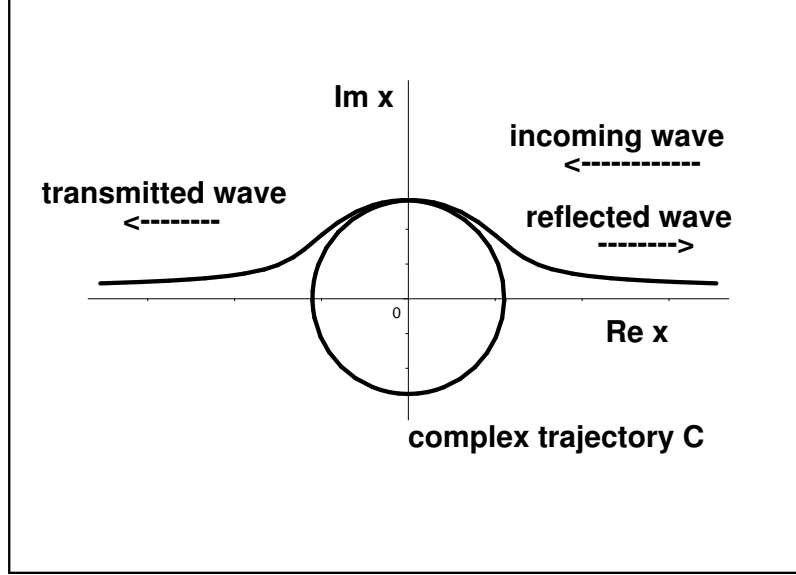


Figure 1: The arrangement of scattering along a loop-shaped complex path which circumscribes the origin and which coincides with the real line at large  $|x|$ .

For the scattering-experiment arrangement as indicated in Fig. 1 we shall postulate that  $x$  (= the argument of the wave function  $\psi(x)$ ) and  $q$  (= the real eigenvalue of a suitable particle-position operator  $\hat{Q}$ ) will coincide for large  $|q| \gg 1$ . Moreover, we may follow Fig. 1 and specify the positive asymptotic domain of  $x \approx q \gg 1$  as the domain of the incoming wave while the large and negative positions  $x \approx q \ll -1$  will be assigned to the outgoing, transmitted wave. Thus, we shall complement Eq. (2) by the scattering boundary conditions

$$\psi(x) = \begin{cases} e^{-i\kappa x} + R e^{i\kappa x}, & x \gg 1, \\ T e^{-i\kappa x}, & x \ll -1 \end{cases} \quad (3)$$

at any energy  $E = \kappa^2$ .

### 3 Scattering dynamical regime

#### 3.1 The reflection and transmission coefficients

The decisive advantage of our choice of  $V(x) = 0$  in Schrödinger Eq. (2) is its exact solvability, at any energy  $E = \kappa^2$ , in terms of Hankel functions [17],

$$\psi(x) = c_1 \sqrt{x} H_\nu^{(1)}(\kappa x) + c_2 \sqrt{x} H_\nu^{(2)}(\kappa x), \quad \nu = \ell + 1/2, \quad x \in \mathcal{C}^{(N)}. \quad (4)$$

The asymptotics of these solutions may easily be derived since at  $|\arg z| < \pi$  and  $\text{Re } \nu > -1/2$  we have [17]

$$\begin{aligned} \sqrt{\frac{\pi z}{2}} H_\nu^{(1)}(z) &= \exp \left[ i \left( z - \frac{\pi(2\nu + 1)}{4} \right) \right] \left( 1 - \frac{\nu^2 - 1/4}{2iz} + \dots \right), \\ \sqrt{\frac{\pi z}{2}} H_\nu^{(2)}(z) &= \exp \left[ -i \left( z - \frac{\pi(2\nu + 1)}{4} \right) \right] \left( 1 + \frac{\nu^2 - 1/4}{2iz} + \dots \right) \end{aligned}$$

In other words, we may abbreviate  $\kappa x = z(x) = z$  and eliminate

$$\begin{aligned} \exp(i \kappa x) &= \sqrt{\frac{\pi \kappa x}{2}} H_\nu^{(1)}(\kappa x) \exp \left[ -i \left( \frac{\pi(2\nu + 1)}{4} \right) \right] \left( 1 - \frac{\nu^2 - 1/4}{2i \kappa x} + \dots \right), \\ \exp(-i \kappa x) &= \sqrt{\frac{\pi \kappa x}{2}} H_\nu^{(2)}(\kappa x) \exp \left[ i \left( \frac{\pi(2\nu + 1)}{4} \right) \right] \left( 1 + \frac{\nu^2 - 1/4}{2i \kappa x} + \dots \right). \end{aligned}$$

These formulae may be inserted in boundary conditions (3) yielding, in the leading order of approximation, the following closed formula for the “far right” wave function  $\psi(x) \approx e^{-i \kappa x} + R e^{i \kappa x} \approx$

$$\approx \sqrt{\frac{\pi \kappa x}{2}} H_\nu^{(2)}(\kappa x) \exp \left[ i \left( \frac{\pi(2\nu + 1)}{4} \right) \right] + R \sqrt{\frac{\pi \kappa x}{2}} H_\nu^{(1)}(\kappa x) \exp \left[ -i \left( \frac{\pi(2\nu + 1)}{4} \right) \right] \quad (5)$$

at  $x \gg 1$ , as well as the complementary asymptotic estimate of the “far left” wave function

$$\psi(x) \approx T e^{-i \kappa x} \approx T \sqrt{\frac{\pi \kappa x}{2}} H_\nu^{(2)}(\kappa x) \exp \left[ i \left( \frac{\pi(2\nu + 1)}{4} \right) \right], \quad x \ll -1. \quad (6)$$

The right-hand-side expression in formula (6) defines in fact a particular *exact* solution of Eq. (2) which may be analytically continued along the *whole* complex integration path  $\mathcal{C} = \mathcal{C}^{(N)}$ . This path, by construction, moves from the left infinity to

the right infinity while performing  $N = -m/2 \geq 1$  clockwise rotations around the origin. This means that at the points belonging to the “far right” part of the curve  $\mathcal{C}^{(N)}$  our function becomes equal to the expression

$$T \sqrt{\frac{\pi z}{2}} H_{\nu}^{(2)}(ze^{im\pi}) \exp \left[ i \left( \frac{\pi(2\nu+1)}{4} \right) \right], \quad x \gg +1. \quad (7)$$

At the same time, such a function has to match the  $x \gg 1$  boundary conditions (3) or (5). Thus, in a way used in Ref. [15] it is now sufficient to recall formula 8.476.7 of ref. [17],

$$H_{\nu}^{(2)}(ze^{im\pi}) = \frac{\sin(1+m)\pi\nu}{\sin\pi\nu} H_{\nu}^{(2)}(z) + e^{i\pi\nu} \frac{\sin m\pi\nu}{\sin\pi\nu} H_{\nu}^{(1)}(z) \quad (8)$$

and to insert it in Eq. (7). By comparing the result with Eq. (5) one arrives at our final explicit formulae

$$R = R(\nu) = \frac{\sin m\pi\nu}{\sin(m+1)\pi\nu} e^{i(4\nu+1)\pi/2} \quad (9)$$

and

$$T = T(\nu) = \frac{\sin\pi\nu}{\sin(m+1)\pi\nu} \quad (10)$$

which characterize the result of the scattering at any topological dynamical-input parameter  $N = -m/2 = 0, 1, \dots$ .

## 3.2 The points of unitarity vs. quantum knots

### 3.2.1 Single loop, $N = 1$ .

In the one-loop arrangement of Fig. 1 with  $N = 1$ ,  $T(\nu) = -1$  and  $|R(\nu)| = 2 \cos \pi\nu$  one arrives at the elementary formula

$$z = z(\nu) = |T|^2 + |R|^2 = 1 + 4 \cos^2 \pi\nu \geq 1. \quad (11)$$

This shows that the unitarity of the scattering is solely being preserved at the integer values of  $\ell = \nu - 1/2$  or, in other words, just in the reflectionless cases. In the context of Ref. [15] it is remarkable to notice that the one-loop quantum knots *also* proved to exist *precisely* at the same values of  $\ell$ .



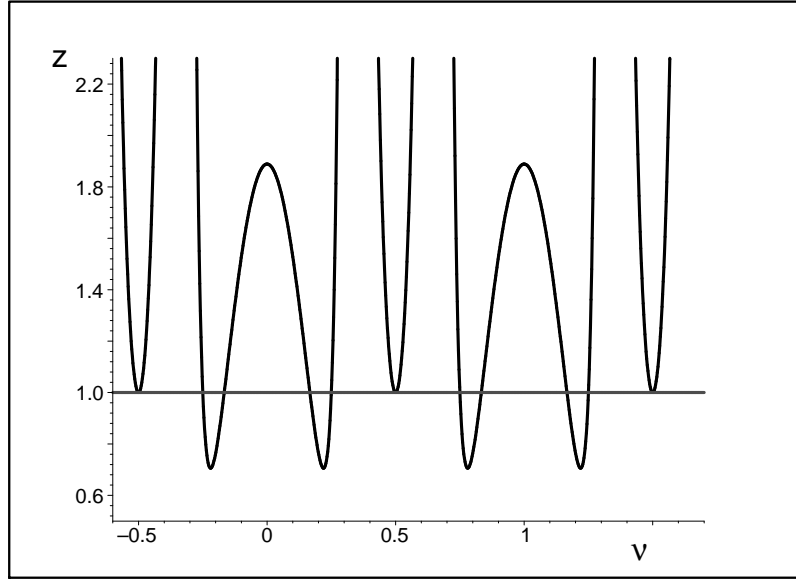


Figure 2: The values of  $z = |T|^2 + |R|^2$  and the  $z = 1$  points of unitarity at  $N = 2$ .

### 3.2.2 Double loop, $N = 2$ .

Starting from the next, two-loop scenario the coefficients  $R$  and  $T$  cease to be bounded. Their trigonometric form remains elementary but the violation of the unitarity acquires a more subtle form (cf. Fig. 2). In particular, there emerge closed intervals of  $\ell$  in which  $z = |T|^2 + |R|^2 \leq 1$  and in which the reflection  $R$  does not vanish.

The above-mentioned remarkable relationship between the incidental  $z = 1$  unitarity of the scattering and the existence of the quantum knots of Ref. [15] merely *partially* survives in the two-loop case. In the interval of  $\nu \in (0, 1)$ , for example, one finds as many as six roots of the unitarity constraint  $z(\nu) = 1$  (including multiplicity, viz., the values of  $\nu = 1/6, 1/4, 1/2, 1/2, 3/4$  and  $5/6$ ). Merely three of them (viz., the values of  $\nu = 1/4, 1/2$  and  $3/4$ ) have been shown to imply the existence of a two-loop quantum knot in Ref. [15].

### 3.2.3 Multiple loops, $N > 2$ .

At the higher winding numbers the overall pattern remains very similar. In particular, the next,  $N = 3$  sample of the function  $z = z(\nu) = |T|^2 + |R|^2$  is given

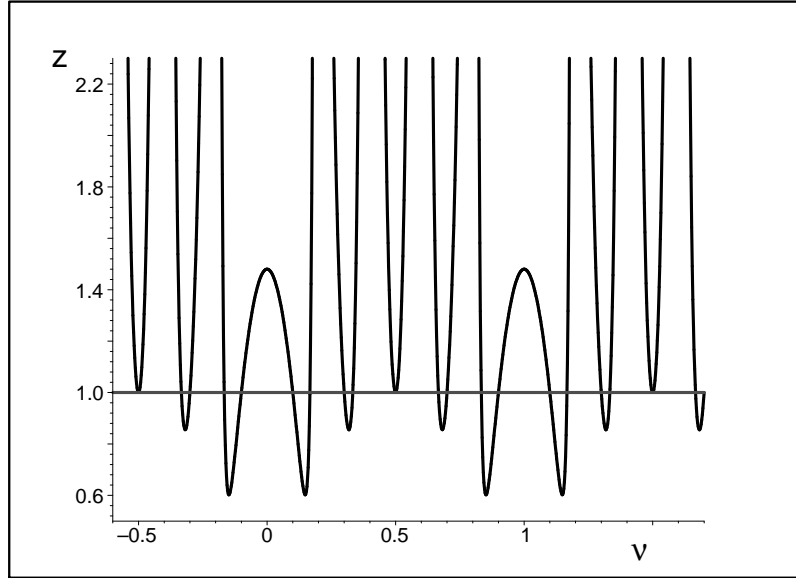


Figure 3: The values of  $z = |T|^2 + |R|^2$  and the  $z = 1$  points of unitarity at  $N = 3$ .

in Fig. 3. The inspection of this picture enables us to reveal that in the interval of  $\nu \in (0, 1)$  there exist five values of  $\nu$  guaranteeing the existence of a three-loop quantum knot [15] (viz.,  $\nu = 1/6, 2/6, 3/6, 4/6$  and  $5/6$ ) but as many as ten roots of equation  $z(\nu) = 1$  (viz., the values of  $\nu = 1/10, 1/6, 3/10, 1/3, 1/2, 1/2, 2/3, 7/10, 5/6$  and  $9/10$ ).

An extrapolation of this pattern to an arbitrary winding number seems straightforward. In a remarkable and nontrivial manner it interrelates the scattering and bound states in the manner which resembles the well known correspondence between the bound states and poles of analytic S-matrix.

## 4 Discussion

### 4.1 General alocal quantum systems

It is well known that the overall, abstract postulates of Quantum Theory do *not* require the observability (i.e., reality) of the one-dimensional coordinate  $x$ . The author of Ref. [18], for example, emphasized that the quantity  $x \in \mathbb{R}$  plays a double role in quantum physics. For the time being let us call them “physical” (A) and

“mathematical” (B). The former role (A) means that whenever our coordinate  $x$  appears as an argument in a wave function,  $\psi = \psi(x)$ , we immediately – and almost always tacitly – assume that, firstly, this wave function describes a point particle moving along a straight line while, secondly, the position of this point particle is measurable and represented by such an operator  $Q$  that  $Q\psi(x) = x\psi(x)$ .

It is important to notice that whenever one starts working with the same-looking wave function  $\psi = \psi(x)$  in scenario (B), there is, first of all, no implication concerning physics. The purely formal reason is that all of the possible concrete realizations of the abstract separable Hilbert space are mutually unitarily equivalent. In this sense, a state of any quantum system may be represented, if we so decide, by a quadratically integrable complex function  $\psi(x) \in \mathbb{L}^2(\mathbb{R}, d\mu)$ . In general, the quantum system in question need not even be assigned any measurable coordinate at all.

In the latter case the meaning of the *real argument* of  $\psi(x)$  may remain *physical* but still *very different* from a spatial position of a localized particle (cf. Ref. [18] for some most elementary illustrative examples). In an extreme alternative, the argument  $x$  of  $\psi(x)$  (defined, in the Dirac’s notation, as equal to a bra-ket overlap,  $\psi(x) = \langle x|\psi \rangle$ ) may even be chosen *complex* (cf. [3] for a nice recent review of some interesting and promising merits of such an option).

## 4.2 Locally alocal systems

Once we decide to work with the entirely formal, Riesz-basis-related concept of the overlaps  $\langle x|\psi \rangle$  which are merely “numbered” by the quantities (or rather “indices”)  $x$  forming, as in our present paper, a left-right symmetric (often called  $\mathcal{PT}$ –symmetric [3]) complex curve  $\mathcal{C}$ , we may still try to generate the time-evolution of the system via a sufficiently simple *ad hoc* Hamiltonian operator  $H$ . One of the most persuasive illustrative example of the appeal of such a model-building direction (admitting that a measurable coordinate does not exist at all) has been offered by Witten [19]. While he tried to understand and/or classify the possible mechanisms of a breakdown of symmetry (called, nowadays, supersymmetry, connecting fermions and bosons) he

proposed an elementary partitioned toy-model Hamiltonian

$$H = \begin{pmatrix} H^{(-)} & 0 \\ 0 & H^{(+)} \end{pmatrix}, \quad H^{(\pm)} = -\frac{d^2}{dx^2} + V^{(\pm)}(x), \quad x \in \mathbb{R} \quad (12)$$

where  $x$  *was not* and observable of course [20].

In a way reemphasized by many other authors [8, 16, 18] we need not really insist on the observability of the coordinate  $x$ , especially when we study bound states. The situation is less liberal in the context of scattering in which one may truly appreciate having the *physical* concept of the *real and measurable* position  $q \in \mathbb{R}^d$ . In this context our present message is that one can weaken the requirement and preserve the concept of the locality *just* in the asymptotic spatial domains.

In this context we introduced here the idea of motion of a quantum (quasi)particle along the path of coordinates of Fig. 1 which are only real (i.e., in principle, observable) asymptotically. Naturally, such an assumption is *sufficient* for the imposition of the “realistic” asymptotic boundary conditions and for the related constructions of certain effective  $\mathcal{PT}$ -symmetric one-dimensional models of scattering by a local potential  $V(x)$  as studied in the recent literature [8, 21].

### 4.3 Open questions

In a way encouraged by the phenomenological as well as methodical success of similar quantum models the key innovation as offered in the present letter lies in the *topological* nontriviality of our present, locally complex trajectories  $\mathcal{C}^{(N)}$ . Such an extension of the perspective remains compatible with the current expectations [21] that the reflection coefficient  $R$  and the transmission coefficient  $T$  will not obey the unitarity constraint  $z = |T|^2 + |R|^2 \neq 1$  in general. Naturally, this freedom opens a number of possible physical interpretations ranging from the theory of open systems up to the possible fructification of the philosophy of the present elementary example in the context of the path-integral quantization or, alternatively, in the recently fashionable context of the models using the concept of a coordinate-dependent mass.

In the language of physics the generic non-unitarity  $z \neq 1$  may be perceived as

a natural consequence of the admitted presence of certain “sources” and “sinks”. These implicit non-Hermitian forces may be intuitively expected to be rooted not only in the imaginary part of the potential but also [22] in the effects caused by the topologically nontrivial deformations of the integration paths as sampled by Eq. (1). This type of connection has been supported here by a rather unexpected observation that the standard connection between bound states and poles of S-matrix might also find its analogue in the present loop-shaped-path context.

In conclusion, one should emphasize that the loop-shape-generated balance between sinks and sources as imposed by  $\mathcal{PT}$ -symmetry is merely long-ranged and, hence, too weak for a reinstallation of the unitarity. Although an overall explanation of this fact does not require any particularly sophisticated mathematics [10], there still exist a few open questions which have been formulated by *physicists*. Jones [8], in particular, imagined that the model-building freedom offered by the *complex local* potentials  $V(x)$  is in fact rather expensive, *both* in terms of the feasibility of the constructions *and* in terms of the appropriate *preparation* of any external physical interaction  $V(x)$ . In this sense, our present “exceptional” choice of  $V(x) = 0$  offered one of particularly efficient ways of circumventing the problem.

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## References

- [1] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541.
- [2] H. Langer and Ch. Tretter, Czechosl. J. Phys. 70 (2004) 1113.
- [3] C. M. Bender, Rep. Prog. Phys. 70 (2007) 947.
- [4] M. Znojil, Phys. Rev. D 78 (2008) 085003.
- [5] M. Znojil, SIGMA 5 (2009) 001.
- [6] M. Znojil, P. Siegl and G. Lévai, Phys. Lett. A 373 (2009) 1921.
- [7] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.
- [8] H. F. Jones, Phys. Rev. D 76 (2007) 125003.
- [9] H. F. Jones, Phys. Rev. D 78 (2008) 065032.
- [10] M. Znojil, Phys. Rev. D 78 (2008) 025026;  
M. Znojil, Phys. Rev. D. 80 (2009) 045009.
- [11] M. Znojil, J. Phys. A: Math. Gen. 39 (2006) 13325.
- [12] M. Znojil, J. Phys. A: Math. Theor. 41 (2008) 215304.
- [13] M. Znojil and H. B. Geyer, Pramana J. Phys. 73 (2009) 299.
- [14] M. Znojil, Phys. Lett. A 374 (2010) 807.
- [15] M. Znojil, Phys. Lett. A 372 (2008) 3591.
- [16] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 1191.
- [17] I. S. Gradshteyn and I. M. Ryzhik, Tablicy integralov, summ, ryadov i proizvedenii. Nauka, Moscow, 1971.
- [18] J. Hilgevoord, Am. J. Phys. 70 (2002) 301.

- [19] E Witten, Nucl. Phys. B188 (1981) 513.
- [20] B. Bagchi, Supersymmetry in Quantum and Classical Mechanics. Chapman and Hall/CRC Press, Boca Raton, 2000.
- [21] F. Cannata, J.-P. Dedonder and A. Ventura, Ann. Phys. (NY) 322 (2007) 397.
- [22] M. Znojil, Phys. Lett. A 342 (2005) 36;  
H. Břila, Pramana - J. Phys. 73 (2009) 307.